Hopf bifurcation cyclicity and chaos for the Z_2-symmetric Rössler system

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Abstract In this paper, the Hopf cyclicity and hidden chaos for the three-dimensional (3D) Z_2-symmetric Rössler system are investigated. Applying the recursive formula of the singular point quantities, and by strict symbolic calculation, we determine the highest order three of weak focus at the symmetric equilibria on center manifold. And under suitable perturbation, six and at most six small amplitude limit cycles can generate from symmetric equilibria via Hopf bifurcation. Furthermore, we study different cases that multiple Hopf bifurcation and chaos can simultaneously occur around the two symmetric equilibria, where one (2, 2) distribution of four limit cycles is accompanied by chaos. To our knowledge, this property is very rare in many chaotic systems.

Keywords Rössler model; multiple Hopf bifurcation; singular point quantities; hidden chaos

MR(2000) Subject Classification 34C23, 34C28, 34C40, 37G15.

1 Introduction

Limit cycle bifurcation, as a very significant topic in the qualitative theory of ODEs, has received considerable attention and exploration. One of its most important research motivations originates from the Hilbert's 16th problem, regarding the maximum number and relative positions of limit cycles in planar differential dynamic systems [1, 2]. Moreover, we consider the following 3D real differential system:

$$\begin{cases} \dot{x} = P_n(x, y, z), \\ \dot{y} = Q_n(x, y, z), \\ \dot{z} = R_n(x, y, z), \end{cases} \tag{1}$$

where the dot represents the first derivative respect to the time t, and P_n, Q_n, R_n are three polynomials with degree n in x, y and z. Then the maximum number and relative positions of limit cycles in the system (1) can be considered as a generalization of the Hilbert's 16th

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problem. As its local problem, Hopf bifurcation of 3D system (1), in most cases only the existence of single limit cycle is investigated, yet for the multiplicity, especially for the Hopf cyclicity, there are only a few works [3]. The main reason lies in the higher difficulty of calculating and determining the highest order focal value at a Hopf bifurcation point [4, 5].

The dimensionality reduction is usually adopted to transform the 3D system (1) into a plane system, and there exist some classical research methods such as Poincaré normal form [6], Lyapunov-Schmidt method [7], the direct calculation of center manifold [8]. the average theory [9], the determination of inverse Jacobi multiplier [10, 11], the simple normal form [12] and the direct calculation of formal first integration [13]. Recently, the authors of [14] proposed the upper bound theorem of the number of limit cycles from a center within the center manifold based on the Bautin ideal, and the upper bounds of the center cyclicity of Lorenz, Chen and Lü systems on the center manifold are given in [15]. Here we will apply the algorithm presented in [16] to calculate the singular point quantities on the center manifold without reducing the dimensionality. Its linear recursive algorithm is both convenient and effective, as is visible in some applications [17, 18, 19].

Here we consider a class of 3D Rössler system with the following form:

$$\begin{cases} \dot{x} = x - xy - z_1, \\ \dot{y} = x^2 - ay, \\ \dot{z_1} = b(cx - z_1), \end{cases} \tag{2}$$

where $a, b, c \in \mathbb{R}$. In fact, in the 1970s, in order to search for different types of chaos presenting in many natural and artificial fields, Rössler proposed a series of models [20, 21]. Later, more wider studies follow them, for example, the authors of [21] classified them through modern topological analysis in 2006. The local and global limit cycles bifurcations of codimension one and two were discussed by Roberto et al. [22]. And in [23, 24], the authors studied respectively the integrability of the classic Rössler system and the four-prototype Rössler system. As for system (2), it was proposed by Rössler in [25], as another model of "Lorenzian chaos" with a surprising type of chaos, i.e., so-called pure cut chaos. Since system (2) has time-preserving symmetry, this is to say, via the following substitutes:

$$(x, y, z_1, t) \mapsto (-x, y, -z_1, t), \tag{3}$$

system (2) is invariant, which can also be called Z_2-symmetric [19, 26].

Recently, the authors of [27] and [28] have respectively applied the time-averaging method and normal form theory to investigate the zero-Hopf bifurcation of model (2). It is worth noting that for almost all classical chaotic models including the Lorenz system, Chen system, Lü system, Chua's system and Lorenz-like (Yang) system, the Hopf bifurcations have been discussed, the readers can see [17, 18, 29, 30] and therein references. However, this problem of system (2) has not been considered from our knowledge. Here we will investigate the multiple Hopf bifurcation and Hopf cyclicity for system (2).

For the coexistence of Hopf bifurcations and chaotic attractor, as far as we know, there exist rare works on this topic. In 2010, Yang and Wei presented the coexistence of chaotic

attractor and two stable node-foci in an autonomous quadratic Lorenz-like chaotic system [31], yet the coexistence of the limit cycles via Hopf bifurcation and chaotic attractors was not introduced in detail. Then in 2017, Wei et al. studied a type of self-exciting homopolar disc dynamo model in [32], they found that this model exhibits hidden chaotic attractors from unstable periodic orbits when generating Hopf bifurcation. Thus, here we will discuss the coexistence from different perspective, so as to better understand and study chaotic behavior of system (2).

The rest of this paper is organized as follows. In the next section, the singular point quantities are calculated for the origin of corresponding complex symmetric system of (2). In Section 3, the weak focus with the highest order 3 is determined for the two symmetric equilibria. Then from this, six and at most six small amplitude limit cycles are obtained at the two symmetric equilibria, namely the Hopf cyclicity is determined. In Section 4, the dynamic property that Hopf bifurcation and chaos occur simultaneously in system (2) is discussed under different parameter conditions, where four limit cycles with (2,2) distribution accompanied by chaos are discovered.

2 Calculation of the singular point quantities

In this section, we will calculate the singular point quantities of the equilibrium. For Rössler model (2), there always exists the equilibrium $O(0,0,0)$, and when assuming $a(1-c) > 0$, we can obtain other two symmetric equilibria of system (2):

$$O_1(\sqrt{a(1-c)}, 1-c, c\sqrt{a(1-c)}) \quad \text{and} \quad O_2(-\sqrt{a(1-c)}, 1-c, -c\sqrt{a(1-c)}).$$

Since system (2) is Z_2-symmetric, when $(x(t), y(t), z_1(t))$ is the solution of (2), then $(-x(t), y(t), -z_1(t))$ is still its solution. Thus, we only choose the equilibrium O_1 to analyze in detail, and there exists the same case for its symmetric equilibrium O_2. In order to facilitate investigating, we transform the equilibrium O_1 to one origin. By setting

$$(x, y, z_1) = (\tilde{x} + \sqrt{a(1-c)}, \tilde{y} + (1-c), \tilde{z}_1 + c\sqrt{a(1-c)}),$$

then system (2) takes the following form:

$$\begin{cases} \dot{x} = cx - (\sqrt{a(1-c)} + x)y - z_1, \\ \dot{y} = (2\sqrt{a(1-c)} + x)x - ay, \\ \dot{z}_1 = b(cx - z_1), \end{cases} \tag{4}$$

where $\tilde{x}, \tilde{y}$ and $\tilde{z}_1$ are still marked as x, y and z_1. Thus, for the investigation of topological properties at the equilibrium O_1 of (2), it is naturally changed into the one at the origin of system (4). The Jacobian matrix at the origin of (4) is

$$J = \begin{pmatrix} c & -\sqrt{a(1-c)} & -1 \\ 2\sqrt{a(1-c)} & -a & 0 \\ bc & 0 & -b \end{pmatrix},$$

and the characteristic equation of matrix J is

$$\lambda^3 + (a+b-c)\lambda^2 + a(b-3c+2)\lambda + 2ab(1-c) = 0. \tag{5}$$

To ensure that system (4) can generate a generic Hopf bifurcation, we need the matrix J to have a pair of pure imaginary eigenvalues $\pm\mathbf{i}\omega$ and one real eigenvalue λ_0, where $\mathbf{i} = \sqrt{-1}$ and $\omega > 0$. Let its corresponding characteristic equation take the following form:

$$(\lambda^2 + \omega^2)(\lambda - \lambda_0) = 0. \tag{6}$$

Therefore, for generating a generic Hopf bifurcation in the neighborhood of the origin of system (4) the critical condition can be obtained as follows:

$$a = \frac{2c + 2bc - b^2 - 3c^2}{b - 3c + 2}, \qquad b \neq 3c - 2, \tag{7}$$

and then

$$\omega^2 = 2c + 2bc - b^2 - 3c^2 > 0, \qquad \lambda_0 = \frac{2b(c-1)}{b-3c+2}. \tag{8}$$

Next, we will standardize the real system (4) at the origin. In fact, by using certain nondegenerate transformation: $(x, y, z_1)' = \tilde{P}\,(x_2, y_2, z_2)'$, system (4) can become the following form:

$$\begin{cases} \dot{x} = -y + P_2(x, y, z_1), \\ \dot{y} = x + Q_2(x, y, z_1), \\ \dot{z}_1 = \lambda_0 z + R_2(x, y, z_1), \end{cases} \tag{9}$$

where $\tilde{P}$ is a nondegenerate real matrix, x_2, y_2 and z_2 are still marked as x, y and z_1, P_2, Q_2 and R_2 are quadratic homogeneous polynomials in x, y and z_1.

Furthermore, from (8) we know that there exist a nondegenerate transformation matrix P, such that matrix J can be diagonalized, i.e.,

$$P^{-1}JP = \begin{pmatrix} \mathbf{i}\omega & & \\ & -\mathbf{i}\omega & \\ & & \lambda_0 \end{pmatrix},$$

and by calculating we obtain the matrix P as follows:

$$P = \begin{pmatrix} \frac{b+\mathbf{i}\omega}{bc} & \frac{b-\mathbf{i}\omega}{bc} & \frac{b-c}{cd_1} \\ -\frac{2\mathbf{i}d_1(b+\mathbf{i}\omega)}{bc(d_1-\mathbf{i}\omega)}\sqrt{\frac{d_0}{d_1}} & \frac{2\mathbf{i}d_1(b-\mathbf{i}\omega)}{bc(d_1+\mathbf{i}\omega)}\sqrt{\frac{d_0}{d_1}} & -\frac{2\omega}{cd_1}\sqrt{\frac{d_0}{d_1}} \\ 1 & 1 & 1 \end{pmatrix},$$

where

$$d_0 = 1 - c, \quad d_1 = b - 3c + 2. \tag{10}$$

Thus, we can directly introduce a nondegenerate transformation: $(x, y, z_1)' = P\,(z, w, u)'$ and rescale the time variable: $t = -\mathbf{i}T/\omega$, make system (4) become the following complex

symmetric form:

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}T} = z + a_{101}uz + a_{011}uw + a_{110}zw + a_{200}z^2 + a_{020}w^2 + a_{002}u^2 = Z, \\ \dfrac{\mathrm{d}w}{\mathrm{d}T} = -(w + b_{011}uz + b_{101}uw + b_{110}zw + b_{020}z^2 + b_{200}w^2 + b_{002}u^2) = -W, \\ \dfrac{\mathrm{d}u}{\mathrm{d}T} = \mathbf{i}d_{001}u + d_{101}uz + d_{011}uw + d_{110}zw + d_{200}z^2 + d_{020}w^2 + d_{002}u^2 = U, \end{cases} \tag{11}$$

where $u \in \mathbb{R}$, $z, w, T \in \mathbb{C}$, and all $a_{kjl}, b_{kjl}, d_{kjl}$ $(kjl = 002, 011, 020, 101, 110, 200)$ can be found in Appendix A. And these coefficients can guarantee that system (9) and (11) are conjugate, namely by means of the substitutes: $z = x + \mathbf{i}y$, $w = x - \mathbf{i}y$, $T = \mathbf{i}\,t$, system (9) can be changed into system (11).

According to theorem 3.1 in [16], we have the recursive formula of calculating the singular point quantities for the origin of (11) as follows.

Theorem 2.1 ([16])**.** *For system* (11)*, when taking* $c_{110} = 1$, $c_{101} = c_{011} = c_{200} = c_{020} = 0$, $c_{nn0} = 0$, $n = 2, 3, \cdots$ *, we can derive successively and uniquely the terms of the following formal series:*

$$F(z, w, u) = zw + \sum_{\alpha+\beta+\gamma=3}^{\infty} c_{\alpha\beta\gamma} z^{\alpha} w^{\beta} u^{\gamma}\,, \tag{12}$$

such that

$$\frac{\mathrm{d}F}{\mathrm{d}T} = \frac{\partial F}{\partial z} Z - \frac{\partial F}{\partial w} W + \frac{\partial F}{\partial u} U = \sum_{m=1}^{\infty} \mu_m (zw)^{m+1}\,, \tag{13}$$

and if $\alpha \neq \beta$ *or* $\alpha = \beta$, $\gamma \neq 0$, $c_{\alpha\beta\gamma}$ *is expressed by the following recursive formula:*

$$c_{\alpha\beta\gamma} = \frac{\sum_{k+j+l=3} \Phi_{kjl}}{\beta - \alpha - \mathbf{i}d_{001}\gamma}\,,$$

and for any positive integer m, μ_m *is expressed by the following recursive formula:*

$$\mu_m = \sum_{k+j+l=3} [\Phi_{kjl}]_{\alpha=\beta=m, \gamma=0},$$

where

$$\Phi_{kjl} = [(\alpha - k + 1)a_{k,j-1,l} - (\beta - j + 1)b_{j,k-1,l} + (\gamma - l)d_{k-1,j-1,l+1}]c_{\alpha-k+1,\beta-j+1,\gamma-l}.$$

Definition 2.1 (see [16])**.** For the origin of system (9) or (11), the μ_m in (13) is called its m-th singular point quantity, and more if all $\mu_m = 0, m = 1, 2, \cdots$, then it is a center on certain center manifold.

Lemma 2.1 (see [16])**.** *For the singular point quantity* μ_m *and the* m*-th focal value* v_{2m+1} *of system* (4) *or* (11)*,* $m = 1, 2, \cdots$*, if* $\mu_1 = \mu_2 = \cdots = \mu_{m-1} = 0$*, then* $v_3 = v_5 = \cdots = v_{2m-1} = 0$ *and* $v_{2m+1} = \mathbf{i}\pi\mu_m$ *hold, namely they are called algebraic equivalent.*

According to the recursive formulas of singular point quantities μ_m in Theorem 2.1, and by using the computer algebra system Mathematica, we can obtain the first three singular point quantities of the origin of system (11) as follows:

$$
\begin{aligned}
\mu_1 &= -\frac{\mathbf{i}\omega d_1}{b^2 c d_0 d_2 d_3} f_1, \\
\mu_2 &= -\frac{\mathbf{i} d_1^2}{3\omega b^5 c^2 d_0^3 d_2^3 d_3^2 d_4} f_2, \\
\mu_3 &= -\frac{\mathbf{i} d_1^3}{72\omega^3 b^8 c^3 d_0^5 d_2^5 d_3^4 d_4^2 d_5 d_6} f_3,
\end{aligned}
\tag{14}
$$

where d_0 and d_1 in (10), and

$$
\begin{aligned}
d_2 = {} & b^4 - 8b^3c + 4b^3 + 20b^2c^2 - 14b^2c - 36bc^3 + 48bc^2 - 16bc + 27c^4 \\
& - 54c^3 + 36c^2 - 8c, \\
d_3 = {} & b^4 - 8b^3c + 4b^3 + 23b^2c^2 - 20b^2c + 3b^2 - 36bc^3 + 48bc^2 - 16bc \\
& + 27c^4 - 54c^3 + 36c^2 - 8c, \\
d_4 = {} & 9b^4 - 72b^3c + 36b^3 + 212b^2c^2 - 190b^2c + 32b^2 - 324bc^3 + 432bc^2 \\
& - 144bc + 243c^4 - 486c^3 + 324c^2 - 72c, \\
d_5 = {} & b^4 - 8b^3c + 4b^3 + 8b^2c^2 + 10b^2c - 12b^2 - 36bc^3 + 48bc^2 - 16bc \\
& + 27c^4 - 54c^3 + 36c^2 - 8c, \\
d_6 = {} & 4b^4 - 32b^3c + 16b^3 + 95b^2c^2 - 86b^2c + 15b^2 - 144bc^3 + 192bc^2 \\
& - 64bc + 108c^4 - 216c^3 + 144c^2 - 32c, \\
f_1 = {} & 4b^6 - 24b^5c + 6b^5 + 23b^4c^2 + 31b^4c - 24b^4 + 117b^3c^3 - 314b^3c^2 \\
& + 219b^3c - 42b^3 - 396b^2c^4 + 936b^2c^3 - 748b^2c^2 + 224b^2c - 16b^2 \\
& + 513bc^5 - 1350bc^4 + 1323bc^3 - 572bc^2 + 92bc - 243c^6 + 729c^5 \\
& - 864c^4 + 504c^3 - 144c^2 + 16c,
\end{aligned}
$$

and

$$
f_2 = \sum_{i=0}^{25} \varphi_i(c)\, b^i, \qquad f_3 = \sum_{i=0}^{56} \psi_i(c)\, b^i,
$$

where all $\varphi_i(c)$ and $\psi_i(c)$ are respectively polynomials only in c, f_2 and f_3 can be found respectively in Appendix B and the website: https://github.com/mathliweiye/Rossler. It should be pointed out that μ_2 is gotten under the condition of $\mu_1 = 0$. Simliarly, μ_3 is gotten under the conditions of $\mu_1 = \mu_2 = 0$.

Remark 2.1. It can be easily verified that the origin of (2) can not be a Hopf bifurcation point under the conditions (7), which guarantees the two symmetric equilibria O_1 and O_2 are Hopf bifurcation points. Thus Hopf bifurcation at the origin is no longer considered.

Furthermore, we obtain the following theorem.

Theorem 2.2. *For system* (9) *or* (11) *restricted to a two-dimensional center manifold, the origin has the first three focal values:*

$$v_3 = \mathbf{i}\pi\mu_1, \qquad v_5 = \mathbf{i}\pi\mu_2, \qquad v_7 = \mathbf{i}\pi\mu_3. \tag{15}$$

In the above expressions, v_{2m+1} *has been reduced under* $v_{2m-1} = 0$*,* $m = 2, 3$*.*

Obviously, under the non-degenerate transformation and conditions (7), the two equilibria O_1 and O_2 of (2) have the same weak focus properties as the origin of (9).

3 The highest order weak focus and Hopf cyclicity

In this section, we mainly discuss two issues: (i) the weak focus reaching its highest order for each symmetric equilibrium O_1 or O_2 of (2); (ii) the maximum number of small amplitude limit cycles bifurcating from each symmetric equilibrium, namely Hopf cyclicity.

3.1 Determination of the highest order weak focus

According to Lemma 2.1, the highest order weak focus can be determined by the first several corresponding singular point quantities. We observe that the first three singular point quantities $\mu_1 = \mu_2 = \mu_3 = 0$ if $d_1 = 0$, i.e., $b = 3c - 2$ holds. However, $d_1 \neq 0$ should be met from the conditions (7) for the existence of Hopf bifurcation point. Thus it is just needed to investigate whether f_1, f_2 and f_3 in (14) can all disappear at the same time. Naturally, at this time, all the factors $b, c, d_0, d_2, d_3, d_4, d_5$ and d_6 are required not to be zero in the denominators of the expressions of μ_1, μ_2 and μ_3. Thus we denote the product of all above related factors, i.e.,

$$f_d = bcd_0d_1d_2d_3d_4d_5d_6. \tag{16}$$

It is noted that f_1, f_2, f_3 in (14) and f_d are all polynomials only with respect to b and c.

To indicate that when $\mu_1 = \mu_2 = 0$, $\mu_3 \neq 0$ holds, here we calculate the resultants of f_d, f_2, f_3 and f_1 with respect to b respectively as follows:

$$\begin{aligned}
\text{Resultant}[f_1, f_d, b] = {} & c^{19}(c-1)^{59}(c-2)(3c-1)(3c-2)^{42}(51c-100) \\
& \times (3c^2 - 14c + 7)(798237c^4 - 1210476c^3 \\
& + 519181c^2 - 3546c - 26112)G_{15}, \\
\text{Resultant}[f_1, f_2, b] = {} & c^{13}(c-1)^{60}(c-2)^2(3c-1)(3c-2)^{46}(51c-100) \\
& \times (3c^2 - 14c + 7)(798237c^4 - 1210476c^3 \\
& + 519181c^2 - 3546c - 26112)^2G_{17}, \\
\text{Resultant}[f_1, f_3, b] = {} & c^{30}(c-1)^{135}(c-2)^3(3c-1)(3c-2)^{100}(51c-100)^2 \\
& \times (3c^2 - 14c + 7)^2(798237c^4 - 1210476c^3 \\
& + 519181c^2 - 3546c - 26112)^2G_{53},
\end{aligned} \tag{17}$$

where $G_{15} = \sum_{i=0}^{15} l_i c^i$, $G_{17} = \sum_{i=0}^{17} m_i c^i$ and $G_{53} = \sum_{i=0}^{53} n_i c^i$ are all polynomials only in c with degree $15, 17$ and 53 respectively, where $l_i, m_i, n_i \in \mathbb{R}$. By observing the three resultants in (17), there exists the product of common factors as follows

$$\begin{aligned} & c(c-1)(c-2)(3c-1)(3c-2)(51c-100)(3c^2-14c+7) \\ & \times (798237c^4 - 1210476c^3 + 519181c^2 - 3546c - 26112) := \tilde{g}. \end{aligned}$$

Then we have the following lemma.

Lemma 3.1. *If $G_{17} = 0$, then $f_1 = f_2 = 0$ can hold, and $f_d f_3 \neq 0$ necessarily.*

Proof. Since $f_d \neq 0$ should be guaranteed when $f_1 = f_2 = 0$, we should exclude the common solutions of $f_1 = f_2 = f_d = 0$. Thus letting

$$\text{Resultant}[f_1, f_2, b] = 0,$$

we have $G_{17} = 0$ or the product $\tilde{g} = 0$, Actually, by computing we obtain

$$\text{Resultant}[G_{17}, \tilde{g}\, G_{15}, c] \neq 0,$$

namely, $G_{17} = 0$ and $\tilde{g}\, G_{15} = 0$ can not hold simultaneously. This implying that the common zeros of f_1 and f_d cannot come from G_{17} but from $\tilde{g}$. This is to say, if $G_{17} = 0$, then $f_d \neq 0$ holds necessarily, and $f_1 = f_2 = 0$ can hold.

Furthermore, by computing we have

$$\text{Resultant}[G_{17}, \tilde{g}\, G_{53}, c] \neq 0.$$

Similarly, the common zeros of f_1 and f_3 cannot come from G_{17} but from $\tilde{g}$. Thus when $G_{17} = 0$, $f_3 \neq 0$ hold necessarily. And by letting $G_{17} = 0$, we can easily find real solutions of $f_1 = f_2 = 0$, which will be given later. From these, we obtain conclusion of the lemma. $\square$

In fact, when $\tilde{g} = f_1 = 0$, it can be verified easily that $f_d = 0$. Thus $\tilde{g} = 0$ is not be considered. Only from $G_{17} = 0$, we can find solutions of $f_1 = f_2 = 0$, then obtain by calculating

$$\begin{aligned} &\text{Resultant}[G_{17}, f_1, c] = H_{17} H_{85}, \\ &\text{Resultant}[G_{17}, f_2, c] = H_{17} H_{408}, \\ &\text{Resultant}[H_{85}, H_{408}, b] \neq 0, \end{aligned}$$

where $H_{17} = \sum_{i=0}^{17} p_i b^i$, $H_{85} = \sum_{i=0}^{85} q_i b^i$ and $H_{408} = \sum_{i=0}^{408} r_i b^i$ are all polynomials only in b with degree $17, 85$ and 408 respectively, where $p_i, q_i, r_i \in \mathbb{R}$. From the above calculation, it can be seen that the solutions of $f_1 = f_2 = 0$ only come from $G_{17} = 0$ and $H_{17} = 0$.

Thus by symbolic computing and the Sturm's theorem [33], we can strictly verify that there exist seven groups of real solutions such that $f_d \neq 0$, $f_1 = f_2 = 0$, $f_3 \neq 0$. At the same time, from (7) and (8), we consider the existence conditions of two symmetric Hopf bifurcation points of system (2), namely must guarantee

$$a(1-c) > 0 \quad \text{and} \quad \omega^2 = 2c + 2bc - b^2 - 3c^2 > 0.$$

Thus only the following two groups of real solutions $(b,c) = (b^{(k)}, c^{(k)}),\ k = 1,2$:

$$\begin{aligned}(b^{(1)}, c^{(1)}) &\approx (-0.1264283230337191,\ 0.1252290887170258),\\ (b^{(2)}, c^{(2)}) &\approx (-0.0987918740205572,\ 0.5889371151553997),\end{aligned}$$

can satisfy the above conditions, this is to say, just the two groups of solutions can make $\mu_1 = \mu_2 = 0,\ \mu_3 \neq 0$ hold respectively.

Therefore, from the lemma 3.1 and the above analysis, we have the following result.

Lemma 3.2. *The singular point quantities $\mu_1 = \mu_2 = 0$ and $\mu_3 \neq 0$ hold if and only if $(b,c) = (b^{(k)}, c^{(k)}),\ k = 1,2$. Thus the origin of system* (9) *or* (11) *can become the weak focus with order 3.*

Subsequently, from the above analysis we know that f_1, f_2 and f_3 cannot disappear simultaneously, and the common factor $d_1 = b - 3c + 2 \neq 0$ in the expressions of μ_1, μ_2 and μ_3 in (14). Thus the first three weak focal values in (15) can not be zero at the same time. And according to Lemma 3.2, we have the following theorem.

Theorem 3.1. *For system* (2) *restricted to certain center manifold, under the conditions* (7) *the two symmetric equilibria O_1 and O_2 are the weak foci which can reach the highest order 3, and cannot become a center.*

Furthermore, applying the above analysis, we can provide the weak focus types, namely in the parameters set of guaranteeing existence of two symmetric Hopf bifurcation points of system (2),

$$\Omega := \{(a,b,c) : a(1-c) > 0,\ a = \frac{\omega^2}{b - 3c + 2},\ \omega^2 = 2c + 2bc - b^2 - 3c^2 > 0\}.$$

we have the following conclusion.

Theorem 3.2. *For system* (2)*, the equilibria $O_{1,2}$ are of order i as the weak foci, if and only if $(a,b,c) \in \mathrm{C}_i,\ i = 1,2,3,$ where*

$$\begin{aligned}\mathrm{C}_1 &:= \{(a,b,c) \in \Omega : f_1 \neq 0, f_d \neq 0\},\\ \mathrm{C}_2 &:= \{(a,b,c) \in \Omega : f_1 = 0, f_2 \neq 0, f_d \neq 0\},\\ \mathrm{C}_3 &:= \{(a,b,c) \in \Omega : (b,c) = (b^{(k)}, c^{(k)}),\ k = 1,2\}.\end{aligned}$$

3.2 Cyclicity of Hopf bifurcation

In this subsection, we will study the cyclicity of Hopf bifurcation at the two symmetric equilibria. From the theorem 3.1, we know that each symmetric equilibrium can be the weak focus reaching its highest order 3, this implies that at most three small amplitude limit cycles are generated at each symmetric equilibrium.

Thus, we mainly determine the existence of three limit cycles via generic Hopf bifurcation by appropriately perturbing the parameters of the original system (2). By setting the characteristic equation (5) with the following form:

$$(\lambda - \delta - \mathbf{i}\omega_\delta)(\lambda - \delta + \mathbf{i}\omega_\delta)(\lambda - \lambda_0) = 0, \tag{18}$$

we can make Jacobian matrix at each symmetric equilibrium have one real root and a pair of conjugate complex roots, i.e.,

$$\lambda_0 = c - a - b - 2\delta \quad \text{and} \quad \lambda_{1,2} = \delta \pm \mathbf{i}\omega_\delta, \tag{19}$$

where $\omega_\delta^2 = a(2 + b - 3c) + 2(a + b - c)\delta + 3\delta^2 > 0$. The necessary condition for the above result (19) is

$$m_2 a^2 + m_1 a + m_0 = 0, \tag{20}$$

where

$$\begin{aligned} m_0 &= -2\delta(b - c + 2\delta)^2, \\ m_1 &= 2c + 2bc - b^2 - 3c^2 - 2(2 + 3b - 5c)\delta - 8\delta^2, \\ m_2 &= 3c - b - 2 - 2\delta. \end{aligned}$$

And compared to the critical condition (7) related to $\delta = 0$, from (20) the disturbing form of coefficient a for system (4) becomes

$$a = \frac{-m_1 - \sqrt{m_1^2 - 4m_0 m_2}}{2m_2} = \frac{2c + 2bc - b^2 - 3c^2}{b - 3c + 2} + O(\delta), \tag{21}$$

where $|\delta| \ll 1$, $m_2 \neq 0$, $m_1^2 - 4m_0m_2 > 0$.

Thus, the corresponding linear system of (9) has the following form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \delta x - y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x + \delta y, \quad \frac{\mathrm{d}z_1}{\mathrm{d}t} = \lambda_0 z_1. \tag{22}$$

Then the related linear form of its conjugate system (11) is changed into

$$\frac{\mathrm{d}z}{\mathrm{d}T} = (1 - \mathbf{i}\delta)z, \quad \frac{\mathrm{d}w}{\mathrm{d}T} = -(1 + \mathbf{i}\delta)w, \quad \frac{\mathrm{d}u}{\mathrm{d}T} = \mathbf{i}d_{001}u. \tag{23}$$

As a matter of fact, all focal values v_{2i-1} are analytic at $\delta = 0$ with respect to parameter δ (see [16]), then the first four focal values can be denoted in the following expessions:

$$\begin{aligned} v_1 &= \mathrm{e}^{2\pi\delta} - 1 = 2\pi\delta + o(\delta), \qquad & v_3 &= \mathbf{i}\pi\mu_1(b,c) + O(\delta), \\ v_5 &= \mathbf{i}\pi\mu_2(b,c) + O(\delta), & v_7 &= \mathbf{i}\pi\mu_3(b,c) + O(\delta). \end{aligned}$$

To determine if there are exact three limit cycles in the neighborhood of each symmetric equilibrium for the original system (2), we need to verify that there exist three small real positive zeros of the Poincaré succession function. This can be done by determined if the Jacobian determinant of the function group (v_1, v_3, v_5) with respect to the variables (δ, b, c) vanishes, see [2] for detail.

From Theorem 2.2 and Lemma 3.2, without loss of generality, we can choose $(0, b^{(1)}, c^{(1)})$ as the critical point of (δ, b, c), and at this time the first four focal values become $v_1 = v_3 = v_5 = 0$ and $v_7 = -209872543090.3246\cdots \neq 0$. Moreover, we obtain the following result of the Jacobian determinant when it is evaluated at the critical point,

$$\det\left[\frac{\partial(v_1, v_3, v_5)}{\partial(\delta, b, c)}\right]_{(\delta,b,c)=(0,b^{(1)},c^{(1)})} = -2\pi \times 6051788561337.3633\cdots \neq 0,$$

this means that there exist certain perturbations of the parameters (δ, b, c) at the critical values such that there are three small real positive zeros of the Poincaré succession function.

Therefore, we can obtain the following theorem.

Theorem 3.3. *Under certain parameter perturbations for system* (2)*, six and at most six small amplitude limit cycles can bifurcate around the symmetric equilibria* O_1 *and* O_2 *with* $(3, 3)$ *distribution via Hopf bifurcation.*

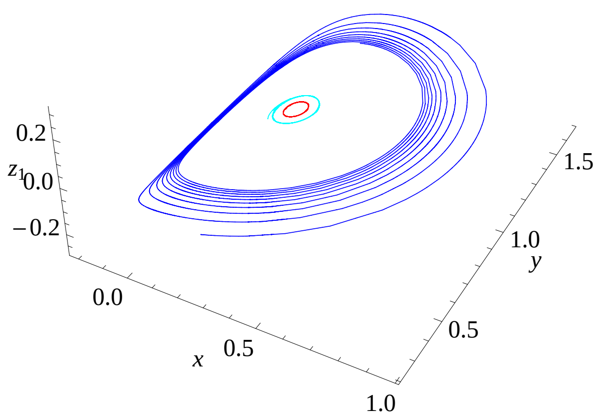


Figure 1: Simulated three limit cycles of system (2) around O_1 in the x-y-z_1 space under the transformation $t \mapsto -t$.

For simulation, we choose $\delta = 0.00001$, $b = -0.1264283330$, $c = 0.1252290787$ as one group of disturbed parameter values at the critical point, then $a \approx 0.1037599644$, yielding the following disturbed four focal values:

$$(v_1, v_3, v_5, v_7) \approx (0.000062832, -0.00015264997, 1.461841649359, -2098726027310).$$

However, from (8) the Jacobian matrix for the origin of (4) has the real eigenvalue

$$\lambda_0 = \frac{2b(c-1)}{b-3c+2} > 0,$$

which implies that the common trajectories no longer approach one center manifold. To be helpful to draw the phase diagram, we make the inverse transformation of time $t \mapsto -t$ on the original system (2). As shown in Figure 1, we provide a phase diagram with three limit cycles around O_1 with the approximate coordinates $(0.301274, 0.874771, 0.037728)$, where two unstable limit cycles which are the largest and the smallest ones, while the middle one is stable limit cycle.

4 The coexistence of Hopf bifurcation and chaos

In this section, based on Theorem 3.2, we will utilize the identified types of the two symmetric equilibria as weak foci to investiate the coexistence of mutiple Hopf bifurcation and chaos

in system (2). For the case C_3, we have conducted corresponding research in the previous section 3, namely under certain small perturbations, three limit cycles can generate around each symmetric equilibrium. Unfortunately, no chaos has been found for C_3. Therefore, we will mainly consider the two cases based on C_1 and C_2. To simplify the discussion, we take $b = 2c$ as a special example to study the two cases separately.

4.1 The case based on C_2

Under the condition $b = 2c$, we have

$$f_1 = c(2 - 3c)(8 - 40c^2 + 19c^3 + 3c^4). \tag{24}$$

And when $f_1 = 0$, we can obtain its four real solutions, but only the following one root satisfies the case C_2:

$$c^* \approx 0.5230551112290062416.$$

Similar to the above analysis for deriving Theorem 3.3, we can induce the perturbation of linear part of system (9) via small changes at the critical values: $c = c^*$ and $\delta = 0$ in the expression (21) of a. It is verified easily that under the critical values condition, the first three focal values $v_1 = v_3 = 0$ and $v_5 = 2678.3819991679 \cdots \neq 0$, and the Jacobian determinant is evaluated approximately as follows

$$\det \left[\frac{\partial(v_1, v_3)}{\partial(\delta, c)} \right]_{(\delta,c)=(0,c^*)} = -2\pi \times 40.4986930817 \cdots \neq 0.$$

These imply that there can exist two small amplitude limit cycles bifurcating from each symmetric equilibrium for system (2). At the same time, the chaos can also occur around the two equilibria O_1 and O_2 of system (2).

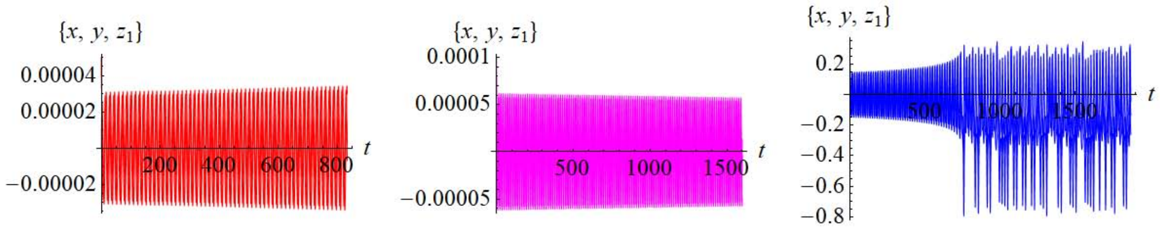
Next, we give an example to illustrate the above interesting dynamic property by numerical simulation. Choosing the following disturbed parameter values

$$\delta = 0.00001, \qquad c = 0.55, \tag{25}$$

yield $a \approx 0.13269017$, $b = 2c = 1.1$ and the first three disturbed focal values as follows:

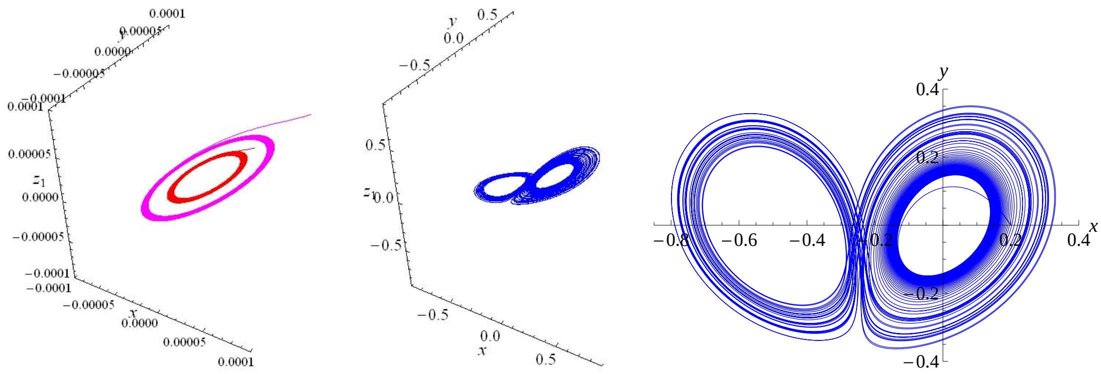
$$(v_1, v_3, v_5) \approx (0.000062831853, -0.948543180911, 2862.45642224).$$

Thus we can obtain two limit cycles around the origin of (4) responding to the equilibrium O_1 of (2). Under the above parameter value conditions, the oscillating time histories of three different initial points are shown in Figure 2(a), (b) and (c), respectively. Additionally, the phase portrait in Figure 3(a) clearly illustrates the existence of two limit cycles, where the inner one is stable and the outer one is unstable. Furthermore, the above example also shows the existence of the chaos, and the simulations are given in Figure 3(b) for the 3D space x-y-z_1 and in Figure 3(c) for the projection plane x-y.



(a) (b) (c)

Figure 2: Simulated time histories of system (4) for $\delta = 0.00001, c = 0.55$ and $b = 2c$ under the initial conditions: (a) $(x, y, z_1) = (0.00005, 0, 0.00005)$; (b) $(x, y, z_1) = (0.0001, 0, 0.0001)$; and (c) $(x, y, z_1) = (0.2, 0, 0.2)$.



(a) (b) (c)

Figure 3: Phase portraits for system (4) with $c = 0.55, b = 2c$ and $\delta = 0.00001$, where the three trajectories correspond in color to the time histories in Figure 2 (a-c) respectively: (a) two limit cycles around the origin where the inner and outer are stable and unstable, respectively; (b) one chaotic attractor in the space x-y-z_1; (c) projection of the chaotic attractor on the plane x-y.

In fact, we apply the fourth-order Runge-Kutta integrator of an adaptive step size, and the classic Wolf algorithm to obtain the three Lyapunov exponents of the chaotic attractor of Rössler system (2) as follows:

$$
\begin{aligned}
L_1 &\approx 0.5499908993, \\
L_2 &\approx -0.1333492692, \\
L_3 &\approx -1.0987917937.
\end{aligned}
$$

Obviously, the maximum Lyapunov exponent $L_1 > 0$, it indicates that system (2) has one chaotic attractor around the symmetric equilibria under the above parameter conditions of (25). Thus we have the following proposition.

Proposition 1. *Under certain coefficient conditions of system* (2)*, four small amplitude limit cycles can bifurcate via the Hopf bifurcation with* $(2, 2)$ *distribution, and one chaos can simultaneously occur around the two symmetric equilibria.*

4.2 The case based on $\mathbf{C}_1$

We consider the case C_1, namely the symmetric equilibria $O_{1,2}$ of system (2) are weak foci of order one. Then its two necessary and sufficient conditions $f_d \neq 0$ and $f_1 \neq 0$ should be met. According to the expressions (16), (24) and (21) of f_d, f_1 and the perturbing a, under the condition of $b = 2c$, by choosing the appropriate values of parameters a and c, we can guarantee the first order weak focal value $v_3 \neq 0$, and the disturbed strong focal value satisfying $0 < |v_1| \approx 2\pi|\delta| \ll |v_3|$ and $v_1 v_3 < 0$.

For example, setting

$$\delta = -0.001, \quad c = \frac{1}{2} \tag{26}$$

yields $a \approx 0.172145$, $b = 2c = 1$ and the first two focal values as follows:

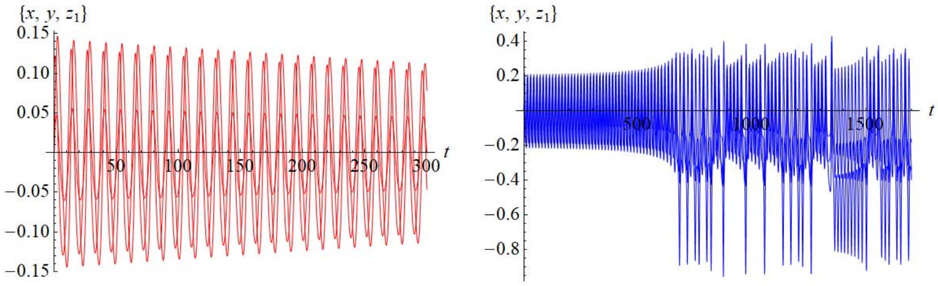
$$(v_1, v_3) \approx (-0.0062831853, \quad 1.0439754049).$$

Thus one unstable limit cycle can be obtained around the origin of (4) responding to the equilibrium O_1 of system (2). At the same time, one chaos also occurs, as depicted in Figure 4 for its simulated time history and in Figure 5 for its phase portraits. And by calculate the three Lyapunov exponents of the chaotic attractor of Rössler system (2), we obtain

$$\begin{aligned} L_1 &\approx 0.4999926964, \\ L_2 &\approx -0.1721412962, \\ L_3 &\approx -0.9987438704, \end{aligned}$$

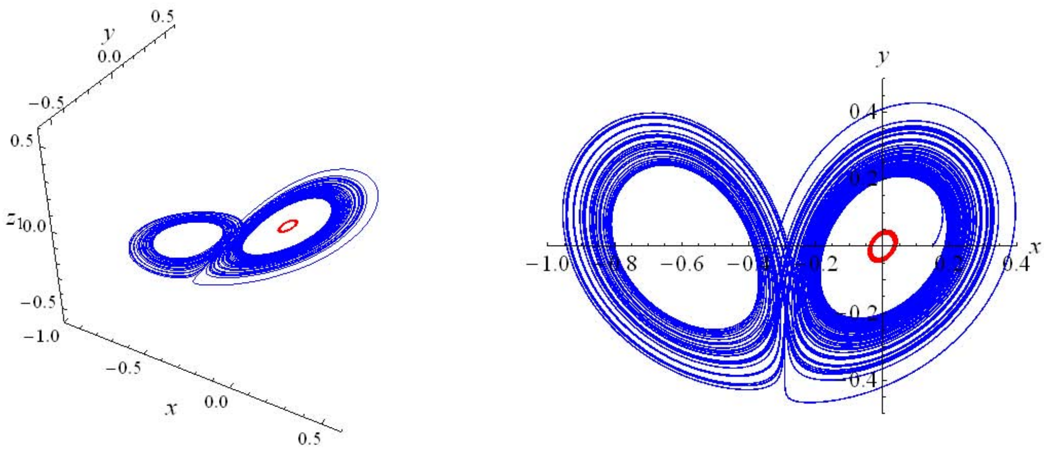
where the maximum Lyapunov exponent $L_1 > 0$, it also verifies that system (2) has one chaotic attractor. Thus we obtain the following proposition.

Proposition 2. *Under certain coefficient conditions of system* (2)*, two small unstable limit cycles can generate via the Hopf bifurcation with* (1, 1) *distribution, and one chaos can simultaneously occur around the two symmetric equilibria.*



(a) (b)

Figure 4: Simulated time history of system (4) for $\delta = -0.001, c = 0.5$ and $b = 2c$ with the initial conditions: (a) $(x, y, z_1) = (0.1, 0, 0)$; and (b) $(x, y, z_1) = (0.15, 0, 0)$.



(a) (b)

Figure 5: (a) Phase portrait of system (4) with $c = 0.5, b = 2c$ and $\delta = -0.001$, where the two trajectories correspond in color to the two time histories in Figure 4 (a-b) respectively: (a) one unstable limit cycle around the origin; and (b) projections of the limit cycle and chaotic attractor on the plane x-y.

Remark 4.1. When δ in (26) becomes a positive small disturbance, the chaos still occurs, but Hopf bifurcation unnecessarily occurs around the origin of system (4). For example, setting $\delta = 0.001$ and still taking $c = \frac{1}{2}, b = 2c = 1$, then at this time we have $a \approx 0.16102665$ and the first two focal values as follows:

$$(v_1, v_3) \approx (0.0062831853, \quad 1.0439754049),$$

then there no longer exists small amplitude limit cycle.

5 Discussion

Applying the algorithm of singular point quantities equivalent to the focal values on center manifold, we investigated the Hopf bifurcation cyclicity and chaos of the Z_2-symmetric Rössler system. Through the discussions of the highest order focal value and center problem, six and at most six small amplitude limit cycles are determined at the two symmetric equilibria with (3,3) distribution. For the coexistence of Hopf bifurcation and chaotic attractor, we have discovered several interesting cases, including the (2,2) distribution and (1,1) distribution of four and two small amplitude limit cycles accompanied by chaos respectively. At the same time, we also find the case that the chaos still occurs without Hopf bifurcation, which implies that there is a correlation between the two, but there is no necessary coexistence.

However, it is still unclear how the coefficients in the original system (2) affect the existence of chaos, for example, the simple disturbance with respect to δ at zero in (21). Therefore, its further research is needed in later work.

Data availability

No data was applied for the research described in the article.

Acknowledgement

This work was partially supported by the Natural Science Foundations of China: No. 12161023 and No. 12061016 (Q. Wang), the Nature Science Foundation of Guangxi: No. 2020GXNSFAA159138 (Q. Wang), Graduate thesis project: No. 2021YJSPY01 (W. Li), and Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation (Q. Wang).

Appendix A. All coefficients $a_{kjl}, b_{kjl}, d_{kjl}$ of system (11) as follows:

$$
\begin{aligned}
a_{200} &= \frac{3d_1(b+\mathbf{i}\omega)^2}{2\omega bc(2bd_0+\mathbf{i}\omega d_1)}\sqrt{\frac{d_0}{d_1}},\\
a_{110} &= \frac{d_1(b+d_1)(d_1+3\mathbf{i}\omega)}{\omega b(d_1+\mathbf{i}\omega)(2bd_0+\mathbf{i}\omega d_1)}\sqrt{\frac{d_0}{d_1}},\\
a_{011} &= \frac{(b^2-2bc+3c-2)(\omega+\mathbf{i}(b-c))}{cd_0d_1(2bd_0+\mathbf{i}\omega d_1)}\sqrt{\frac{d_0}{d_1}},\\
a_{101} &= \frac{(b+\mathbf{i}\omega)(3b^2-10bc+4b+9c^2-6c-\mathbf{i}\omega d_1)}{\omega cd_1(2bd_0+\mathbf{i}\omega d_1)}\sqrt{\frac{d_0}{d_1}},\\
a_{020} &= \frac{d_1(b-\mathbf{i}\omega)^2(d_1-\mathbf{i}\omega)(b^2-2b+3c-2+\mathbf{i}\omega d_1)}{2\omega bcd_0(b+d_1)(d_1+\mathbf{i}\omega)(2bd_0+\mathbf{i}\omega d_1)}\sqrt{\frac{d_0}{d_1}},\\
a_{002} &= \frac{b(b-c)(d_1-\mathbf{i}\omega)(d_1^2(b-c)+\mathbf{i}\omega(b^2+4bc-6b-9c^2+18c-8))}{4\omega cd_0d_1^2(b+d_1)(2bd_0+\mathbf{i}\omega d_1)}\sqrt{\frac{d_0}{d_1}},\\
d_{001} &= \frac{2bd_0}{\omega d_1},\\
d_{002} &= -\frac{3\mathbf{i}b(b-c)^2}{cd_2}\sqrt{\frac{d_0}{d_1}},\\
d_{110} &= \frac{2\mathbf{i}d_1^2(b^2-2bc+3c-2)}{bd_0d_2}\sqrt{\frac{d_0}{d_1}},\\
d_{020} &= -\frac{d_1^2(b-\mathbf{i}\omega)^2(3b^2-10bc+4b+9c^2-6c+\mathbf{i}\omega d_1)}{\omega bcd_2(d_1+\mathbf{i}\omega)}\sqrt{\frac{d_0}{d_1}},\\
d_{011} &= -\frac{2d_1(b-c)(b-\mathbf{i}\omega)(3b^2-8bc+2b+9c^2-6c+2\mathbf{i}\omega d_1)}{\omega cd_2(d_1+\mathbf{i}\omega)}\sqrt{\frac{d_0}{d_1}},\\
d_{200} &= -\bar{d}_{020}, \quad d_{101} = -\bar{d}_{011},\\
b_{kjl} &= \bar{a}_{kjl}\ (kjl = 002, 011, 020, 101, 110, 200).
\end{aligned}
$$

Appendix B. The f_2 in (14) has the following form.

$f_2 = 108b^{25} - 54(44c-1)b^{24} + 3(6023c^2+3443c-1582)b^{23} + 3(3413c^3-93583c^2+36356c+3306)b^{22} + (-1434249c^4+3476702c^3-728100c^2-903759c+272506)b^{21} + (15853326c^5 -$

$28810627c^4 - 749263c^3 + 19249602c^2 - 8901686c + 1063432)b^{20} + (-112065057c^6 + 195926944c^5 + 21388015c^4 - 194509501c^3 + 118270511c^2 - 24285448c + 1203844)b^{19} + (616813185c^7 - 1217273544c^6 + 173827252c^5 + 1027023884c^4 - 812218076c^3 + 211424951c^2 - 9485084c - 2053156)b^{18} + (-2838182319c^8 + 6933236795c^7 - 4443481123c^6 - 1242877532c^5 + 2151550595c^4 - 401310034c^3 - 218911270c^2 + 86154224c - 7489720)b^{17} + (11229394014c^9 - 34710213871c^8 + 41326380463c^7 - 26440606934c^6 + 14749181166c^5 - 10335400084c^4 + 5587973340c^3 - 1636614680c^2 + 216331272c - 8488816)b^{16} + (-38560017645c^{10} + 148182076090c^9 - 251299936641c^8 + 267366624780c^7 - 216456703945c^6 + 139933297404c^5 - 66034958255c^4 + 20046012680c^3 - 3411832908c^2 + 256764560c - 3674816)b^{15} + (115140976077c^{11} - 534224480811c^{10} + 1145065846565c^9 - 1546778249871c^8 + 1486211516675c^7 - 1046143203179c^6 + 525080174471c^5 - 177603142005c^4 + 37276954394c^3 - 4213079468c^2 + 181429768c + 443392)b^{14} - 2(149453423001c^{12} - 814129053570c^{11} + 2073142468463c^{10} - 3288784273509c^9 + 3601148440290c^8 - 2813098996518c^7 + 1562153870049c^6 - 600423359894c^5 + 152028523780c^4 - 23239620904c^3 + 1795558208c^2 - 40930944c - 432128)b^{13} + (673996286331c^{13} - 4212901260507c^{12} + 12302917750659c^{11} - 22149569547649c^{10} + 27150706447361c^9 - 23557054538695c^8 + 14595975487167c^7 - 6388988106903c^6 + 1918537440844c^5 - 374053753776c^4 + 42713239744c^3 - 2282507856c^2 + 25331968c+196608)b^{12} - c(3c-2)(439245306495c^{12} - 2801772724386c^{11} + 8271815193461c^{10} - 14818480675800c^9 + 17721062499993c^8 - 14688627655474c^7 + 8512997488661c^6 - 3409923706570c^5 + 913492778660c^4 - 153536702456c^3 + 14309232784c^2 - 560435232c + 3469824)b^{11} + 2c(3c-2)^2(123693962409c^{12} - 803905728252c^{11} + 2383510542418c^{10} - 4212987370675c^9 + 4879189712060c^8 - 3844353664757c^7 + 2078029383682c^6 - 760068244667c^5 + 181037854184c^4 - 26028488218c^3 + 1944348636c^2 - 53746384c + 265216)b^{10} - 2c(3c-2)^3(59871846357c^{12} - 396130665927c^{11} + 1173958922132c^{10} - 2036321430490c^9 + 2273915212777c^8 - 1697935063782c^7 + 853993106577c^6 - 284345485926c^5 + 59842321964c^4 - 7257365816c^3 + 420621680c^2 - 7631296c + 24576)b^9 + 2c^2(3c-2)^4(24690954456c^{11} - 166331275989c^{10} + 491224001264c^9 - 833407346639c^8 + 895053389032c^7 - 632165403301c^6 + 295197027805c^5 - 89099335907c^4 + 16411412622c^3 - 1641165626c^2 + 69649648c - 739264)b^8 - c^2(3c-2)^6(5710267548c^{10} - 35419997061c^9 + 91663283149c^8 - 129862919916c^7 + 110436357183c^6 - 57846265264c^5 + 18317131577c^4 - 3272367764c^3 + 282125040c^2 - 8171792c + 38144)b^7 + c^3(3c-2)^7(1635904818c^9 - 10410697614c^8 + 26695330231c^7 - 36597385925c^6 + 29499197907c^5 - 14307136359c^4 + 4058165635c^3 - 613298001c^2 + 39579212c - 616244)b^6 - c^3(3c-2)^9(125604999c^8 - 742554552c^7 + 1607618162c^6 - 1719743179c^5 + 987400168c^4 - 299367564c^3 + 42436546c^2 - 1979608c + 11224)b^5 + c^4(3c-2)^{10}(22344345c^7 - 139158624c^6 + 297723910c^5 - 304622119c^4 + 162296099c^3 - 43641547c^2 + 4967299c - 127955)b^4 - c^4(3c-2)^{12}(959022c^6 - 5835564c^5 + 9877256c^4 - 6922659c^3 + 2073168c^2 - 210272c + 1917)b^3 + 3c^5(3c-2)^{13}(26451c^5 - 188448c^4 + 314030c^3 - 205280c^2 + 53998c - 4099)b^2 - 3c^5(3c-2)^{15}(351c^4 - 3345c^3 + 3691c^2 - 1118c + 61)b - 27c^6(3c-2)^{17}(3c-1).$

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